Control Systems : Set 2 : PID (1) - Solutions

Prob 1 | The dynamics of a DC-motor is described by the differential equation

$$\dot{y} + 60y = 600v_a - 1500w$$

where y is the motor speed, v_a is the armature voltage, and w is the load torque. Assume the armature voltage is computed using the PI control law

$$v_a = k_p e + k_I \int_0^t e dt$$

where $e = r - y$ for the reference speed r.

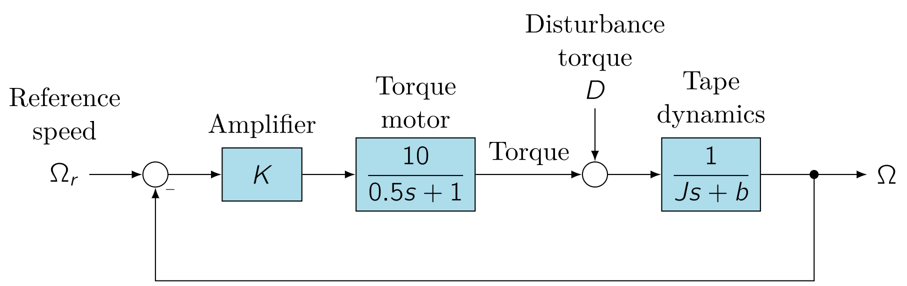
a) Compute the transfer function from W to Y as a function of k_p and k_I

> Compute the closed-loop transfer function, assuming $R = 0$
>
> $$\begin{aligned} (s+60)Y &= 600V_a - 1500W \\ (s+60)Y &= 600k_p(-Y) + 600k_I\frac{1}{s}(-Y) - 1500W \\ s(s+60)Y &= -600k_p sY - 600k_I Y - 1500Ws \\ (s(s+60) + 600k_p s + 600k_I)Y &= -1500Ws \\ \frac{Y}{W} &= \frac{-1500s}{s^2 + (60+600k_p)s + 600k_I} \end{aligned}$$

b) Compute values for k_p and k_I so that the characteristic equation of the closed-loop system will have roots at $-60 \pm 60j$.

> We want the characteristic equation $c(s) = (s+60+60j)(s+60-60j) = s^2 + 120s + 7200$
>
> We equate the coefficients from the previous question to get
>
> $$\begin{aligned} 60 + 600k_p &= 120 & \rightarrow & \quad k_p = 0.1 \\ 600k_I &= 7200 & \rightarrow & \quad k_I = 12 \end{aligned}$$

Prob 2 | A speed control system for a magnetic tape-drive is shown in the figure below, where the constants are $J = 0.10 kg \cdot m^2$ and $b = 1.00 N \cdot m \cdot sec$.



a) Assuming the reference is zero, what is the steady-state error due to a step disturbance torque of $1Nm$? What must the amplifier gain K be in order to achieve a steady-state error $e_{ss} \leq 0.01$rad/sec?

We first compute the closed-loop transfer function from the disturbance to the output

$$\begin{aligned}
\Omega &= \frac{1}{Js+b}\left(D + \frac{10}{0.5s+1}K(-\Omega)\right) \\
(Js+b)(0.5s+1)\Omega &= D(0.5s+1) - 10K\Omega \\
((Js+b)(0.5s+1)+10K)\Omega &= (0.5s+1)D \\
\frac{\Omega}{D} &= \frac{0.5s+1}{(Js+b)(0.5s+1)+10K} \\
&= \frac{0.5s+1}{(0.1s+1)(0.5s+1)+10K}
\end{aligned}$$

The error $E = \Omega_r - \Omega$ is just $-\Omega$, as we've assumed $\Omega_r = 0$.

Assuming that K is chosen so that the system is stable, we can use the Final Value Theorem to compute the steady-state error in response to the unit step input $D = \frac{1}{s}$

$$\begin{aligned}
\lim_{s\to 0} sE(s) &= \lim_{s\to 0} -s\frac{0.5s+1}{(0.1s+1)(0.5s+1)+10K}\frac{1}{s} \\
&= \frac{-1}{1+10K}
\end{aligned}$$

To achieve an error less than 0.01rad/sec, we choose

$$\frac{1}{1+10K} \leq \frac{1}{100} \qquad \rightarrow \qquad K = 9.9 \approx 10$$

(Note that we dropped the negative sign here, as we only care about the magnitude of the error here)

b) Give the damping ratio and the natural frequency of the closed-loop system from Ω_r to Ω, and sketch the time response of the output for a step reference input using the gain K computed in the previous part. Is this a good control system?

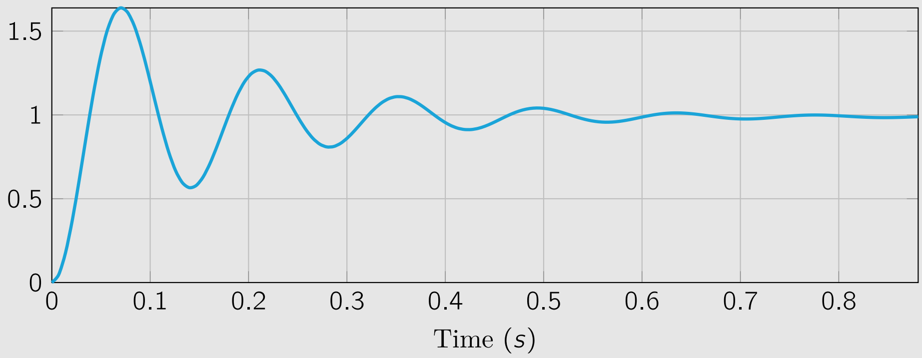
The closed-loop transfer function is

$$\begin{aligned}
\Omega &= \frac{1}{Js+b}\frac{10K}{0.5s+1}(\Omega_r - \Omega)\\
((Js+b)(0.5s+1)+10K)\Omega &= 10K\Omega_r\\
\frac{\Omega}{\Omega_r} &= \frac{10K}{(Js+b)(0.5s+1)+10K}\\
&= \frac{10K}{(0.1s+1)(0.5s+1)+10K}\\
&= \frac{200K}{s^2+12s+(200K+20)}\\
&= \frac{2000}{s^2+12s+2020} && \text{For } K=10
\end{aligned}$$

We can now compute the natural frequency and damping ratio

$$\omega_n = \sqrt{2020} = 45$$
$$\zeta = \frac{12}{2\cdot 45} = 0.13$$

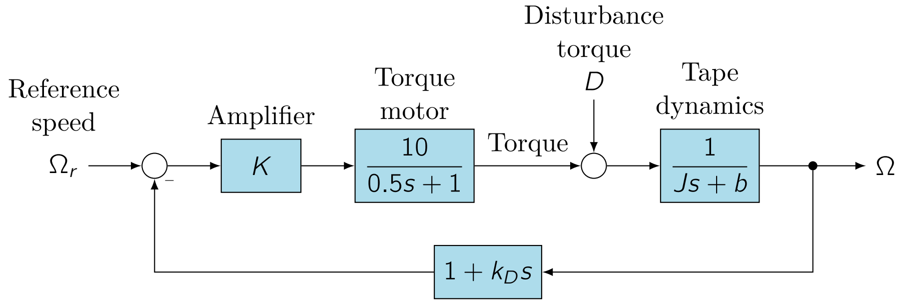
We can sketch the time response by first computing the damped frequency, the peak time, the overshoot, the settling time and the DC gain. These numbers allow us to sketch an appropriate curve, as seen below.



We can conclude that this is not a very good controller, as we have around 60% overshoot and a lot of oscillation.

c) Give values for K and k_D for a PD controller which will meet the specifications of a 1% settling time of $t_s \leq 0.1$sec and an overshoot $M_p \leq 5\%$.

Note: Recall that we don't take the derivative of the reference, and so we place the derivative term in the feedback path as shown below.



> The transfer function is now
>
> $$\begin{aligned}\Omega &= \frac{1}{0.1s+1}\frac{K10}{0.5s+1}\left(\Omega_r - (1+k_D s)\Omega\right)\\ ((0.1s+1)(0.5s+1)+10K(1+k_D s))\Omega &= 10K\Omega_r\\ \frac{\Omega}{\Omega_r} &= \frac{10K}{(0.1s+1)(0.5s+1)+10K(1+k_D s)}\\ &= \frac{200K}{s^2+(200Kk_D+12)s+200K+20}\end{aligned}$$
>
> Where we can see that by selecting K and k_D appropriately, we can choose any desired damping ratio and natural frequency.
>
> To meet the specification, we could choose
>
> $$\begin{aligned} t_s \le 0.1 &\quad \rightarrow \quad \sigma \ge 46\\ M_p \le 0.05 &\quad \rightarrow \quad \zeta \ge 0.7\end{aligned}$$
>
> and then solve to compute K and k_D
>
> $$\begin{aligned}\omega_n^2 = \frac{\sigma^2}{\zeta} = 4,318 = 200K+20 &\quad \rightarrow \quad K = 21.5\\ 2\sigma = 200Kk_D + 12 = 92 &\quad \rightarrow \quad k_D = 0.0186\end{aligned}$$

d) How would the disturbance-induced steady-state error change with the new control scheme in the previous part? How could the steady-state error to a disturbance torque be eliminated entirely?

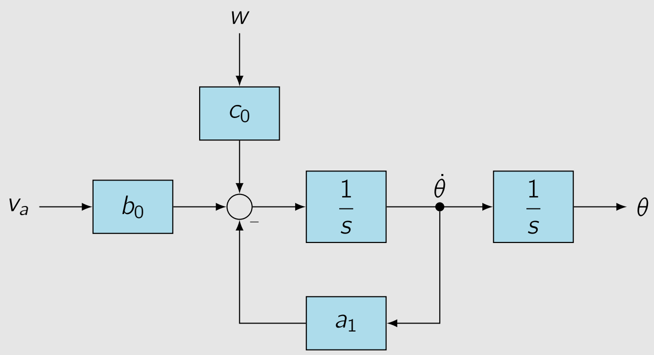
> The derivative term of the controller only impacts the transient response, and so the steady-state behaviour will be the same. To eliminate the steady-state offset, we need to add an integrator to the controller.

Prob 3 | A linear ODE model of a DC motor with negligible armature inductance and with a disturbance torque w is given by

$$\ddot{\theta} + a_1\dot{\theta} = b_0 v_a + c_0 w$$

where θ is the position of the motor and is measured in radians, v_a is the applied voltage in Volts, w is the load torque and a_1, b_0 and c_0 are motor-dependent constants.
With rotating potentiometers, it is possible to measure the positioning error between θ and the reference angle θ_r or $e = \theta_r - \theta$. With a tachometer we can measure the motor speed $\dot{\theta}$.

a) Draw a block diagram of the resulting feedback system showing both θ and $\dot{\theta}$ as variables in the diagram representing the motor.

> The diagram can be created directly from the ODE expression.
>
> 
>

b) Suppose the motor constants are $a_1 = 65$, $b_0 = 200$, and $c_0 = 10$. If there is no load torque ($w = 0$), what speed (in rpm) results from $v_a = 100V$?

> We could compute Laplace transforms and use the final value theorem, or we could just realize that since the system is in steady state, the acceleration must be zero $\ddot{\theta} = 0$.
>
> $$\dot{\theta} = \frac{b_0}{a_1} v_a = \frac{200}{65} 100 = 307.7 \frac{rad}{sec} = 307.7 \cdot 60 \frac{sec}{min} \cdot \frac{1}{2\pi} \frac{revolutions}{rad} \frac{rad}{sec} = 2,938.3\text{rpm}$$

c) Using the parameter values given in the previous part, consider the feedback controller on the error $e = \theta_r - \theta$ and the motor speed $\dot{\theta}$ in the form

$$v_a = k_p e - k_D \dot{\theta}$$

Select the controller parameters k_p and k_D such that the response to a step input in the reference has approximately 17% overshoot and settles within 5% of steady-state in less than 0.05 seconds, when the disturbance is zero $w = 0$.

> The relationship between v_a and θ is
>
> $$s^2\theta + 65s\theta = 200v_a$$

Replacing v_a with out control law gives the closed-loop transfer function

$$
\begin{aligned}
s^2\theta + 65s\theta = 200v_a &= 200(k_p(\theta_r - \theta) - k_D s\theta) \\
\theta(s^2 + 65s + 200k_p + 200k_D s) &= 200k_p\theta_r \\
\frac{\theta}{\theta_r} &= \frac{200k_p}{s^2 + (65 + 200k_D)s + 200k_p}
\end{aligned}
$$

To select our desired damping ratio and natural frequency

$$
\begin{aligned}
M_p \le 17\% \quad &\Rightarrow \quad \zeta \ge -\frac{\ln M_p}{\sqrt{\ln(M_p)^2 + \pi^2}} = 0.49 \\
T_s \le 0.05 \quad &\Rightarrow \quad \sigma \ge \frac{-\ln\delta}{T_s} = \frac{3}{0.05} = 60
\end{aligned}
$$

We can now solve for our controller parameters

$$
\begin{aligned}
65 + 200k_D = 2\sigma = 120 \quad &\Rightarrow \quad k_D = 0.275 \\
200k_p = \omega_n^2 = \frac{\sigma^2}{\zeta} = 14,993 \quad &\Rightarrow \quad k_p = 75
\end{aligned}
$$